https://www.linkedin.com/feed/update/urn:li:activity:6548064949242077184

Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_n}{n!}=a>0$.

Find $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)$.

Solution by Arkady Alt , San Jose ,California, USA.

Let $b_n := \frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}$. Since $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$ and $\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n!}}=1$

then $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n}}(b_n-1)=$

$\lim_{n\to\infty}\left(n\cdot\frac{1}{\sqrt[n]{\frac{a_n}{n!}}}\cdot\frac{n}{\sqrt[n]{n!}}\cdot(b_n-1)\right)=e\lim_{n\to\infty}n\cdot(b_n-1)$.

Noting that $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{\sqrt[n]{\frac{a_n}{n!}}}{\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}\cdot\frac{\frac{n+1}{\sqrt[n+1]{(n+1)!}}}{\frac{n}{\sqrt[n]{n!}}}\right)=$

$\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n!}}}{\lim_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}}\cdot\frac{\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}}{\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}}=1$ we obtain that

$\lim_{n\to\infty}n\cdot(b_n-1)=\lim_{n\to\infty}\left(n\ln b_n\cdot\frac{e^{\ln b_n}-1}{\ln b_n}\right)=\lim_{n\to\infty}n\ln b_n$ (because $\lim_{n\to\infty}\ln b_n=0$).

Since $b_n^n=\frac{(n+1)^{2n}}{n^{2n}}\cdot\frac{a_n}{a_{n+1}}\cdot\sqrt[n+1]{a_{n+1}}=$

$\frac{(n+1)^{2n}}{n^{2n}}\cdot\frac{a_n/n!}{a_{n+1}/(n+1)!}\cdot\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot(n+1)=$

$\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{a_n/n!}{a_{n+1}/(n+1)!}\cdot\sqrt[n+1]{\frac{a_{n+1}}{(n+1)!}}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}$ then

$\lim_{n\to\infty}b_n^n=e^2\cdot\frac{a}{a}\cdot 1\cdot\frac{1}{e}=e$ and, therefore, $\lim_{n\to\infty}(n\ln b_n)=\ln\left(\lim_{n\to\infty}b_n^n\right)=\ln e=1$.

Hence, $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=e\lim_{n\to\infty}n\cdot(b_n-1)=e\lim_{n\to\infty}n\ln b_n=e$.